

THE FREE BOUNDARY OF VARIATIONAL INEQUALITIES WITH GRADIENT CONSTRAINTS

MOHAMMAD SAFDARI

ABSTRACT. In this paper we prove that the free boundary of the minimizer of

$$I(v) := \int_U \frac{1}{2} |Dv|^2 - \eta v \, dx,$$

subject to the pointwise gradient constraint

$$|Dv|_p \leq 1,$$

is as regular as the tangent bundle of the boundary of the domain. To this end, we study a generalized notion of ridge of a domain in the plane, which is the set of singularity of the distance function in the p -norm to the boundary of the domain.

1. INTRODUCTION

Let U be a bounded, open set in \mathbb{R}^2 whose boundary is at least $C^{2,1}$.¹ Suppose $K \subset \mathbb{R}^2$ is a balanced (symmetric with respect to the origin) closed convex set whose interior contains the origin. Let

$$\gamma_K(x) := \inf\{\lambda > 0 \mid x \in \lambda K\}$$

be the gauge function of K . Also let

$$K^\circ := \{x \mid x \cdot k \leq 1 \text{ for all } k \in K\}$$

be the polar of K . By our assumptions, γ_K and γ_{K° are norms on \mathbb{R}^2 (see Rockafellar [11]). Let d_{K° be the metric associated to the norm γ_{K° .

We assume that γ_{K° is strictly convex. This is equivalent to requiring K° to be strictly convex, i.e. its boundary does not contain any line segment. The latter happens, for example, when K is strictly convex and its boundary is C^1 (see Crasta and Malusa [6]). As an example, let $K = \{x \mid \gamma_q(x) \leq 1\}$ for $q > 1$, where $\gamma_q(x) = |x|_q := (|x_1|^q + |x_2|^q)^{1/q}$ is the q -norm of $x = (x_1, x_2)$. Then $\gamma_{K^\circ} = \gamma_p$ is strictly convex, where $p = \frac{q}{q-1}$ is the dual exponent to q .

Let

$$I(v) := \int_U \frac{1}{2} |Dv|^2 - \eta v \, dx,$$

where $\eta > 0$. Let u be the minimizer of I over

$$W_K := \{v \in H_0^1(U) \mid \gamma_K(Dv) \leq 1 \text{ a.e. } \}.$$

It can be shown that u is also the minimizer of I over

$$\{v \in H_0^1(U) \mid v(x) \leq d_{K^\circ}(x, \partial U) \text{ a.e. } \}.$$

For the proof see Brezis and Sibony [1], Treu and Vornicescu [13] and Safdari [12].

¹This implies that U is the union of finitely many components, and each component is the interior of a simple closed Jordan curve with finitely many holes. Each hole is itself, the interior of a simple closed Jordan curve.

When K is the unit disk, and therefore $\gamma_K, \gamma_{K^\circ}$ are both the Euclidean norm, the above problem is the famous elastic-plastic torsion problem. The regularity of the free boundary of elastic-plastic torsion problem is studied by Caffarelli and Rivière [3, 4], Caffarelli and Friedman [2], Friedman and Pozzi [9], and Caffarelli, Friedman, and Pozzi [5]. Their work is explained by Friedman [8].

In this paper, we extend their results to the more general problem explained above. A motivation for our study was to fill the gap between the known regularity results mentioned above and the still open question of regularity of the minimizer of some convex functionals subject to gradient constraints arising in random surfaces. To learn about the latter see the work of De Silva and Savin [7].

In order to study the free boundary, we generalize the notion of ridge, by replacing the Euclidean norm by other norms. We also consider the singularities of the distance function (in the new norm) to the boundary of a domain. These notions have been considered and applied before by Li and Nirenberg [10], and Crasta and Malusa [6]. In particular the work of Crasta and Malusa [6] has considerable intersection with ours. The difference between our work and theirs lies in that we allow less regular domains, and consider norms that are less restricted in some aspects than what they consider.

2. THE RIDGE²

First, we start by generalizing the notion of ridge.

Definition 1. The K -ridge of U is the set of all points $x \in U$ where

$$d_K(x) := d_K(x, \partial U)$$

is not $C^{1,1}$ in any neighborhood V of x . We denote it by

$$R_K.$$

Lemma 1. Suppose γ_K is strictly convex. If $d_K(x) = \gamma_K(x - y) = \gamma_K(x - z)$ for two different points y, z on ∂U , then d_K is not differentiable at x .

Proof. Along the segment \overline{xy} (and similarly \overline{xz}) we have

$$\begin{aligned} d_K\left(x + \frac{t}{\gamma_K(x - y)}(y - x)\right) &= \gamma_K\left(x + \frac{t}{\gamma_K(x - y)}(y - x) - y\right) \\ (2.1) \qquad \qquad \qquad &= \gamma_K(x - y) - t. \end{aligned}$$

Now suppose to the contrary that d_K is differentiable at x , then we have

$$(2.2) \qquad Dd_K(x) \cdot \frac{y - x}{\gamma_K(x - y)} = -1 = Dd_K(x) \cdot \frac{z - x}{\gamma_K(x - z)}.$$

We know that $\gamma_{K^\circ}(Dd_K(x)) = 1$. Thus, by strict convexity of γ_K , there is at most one direction e with $\gamma_K(e) = 1$ along which

$$D_e d_K(x) = Dd_K(x) \cdot e = -1.$$

Therefore d_K can not be differentiable at x . □

Definition 2. The subset of the K -ridge consisting of the points with more than one d_K -closest point on ∂U , is denoted by $R_{K,0}$.

Now we return to our minimization problem. We assume that the minimizer, u , is in $C^1(U)$. See Safdari [12] for the proof of a stronger regularity assuming some extra restrictions on γ_{K° .

²In this section we do not need any assumptions about ∂U .

Lemma 2. *We have*

$$\{x \in U \mid u(x) = d_{K^\circ}(x)\} = \{x \in U \mid \gamma_K(Du(x)) = 1\},$$

and

$$\{x \in U \mid u(x) < d_{K^\circ}(x)\} = \{x \in U \mid \gamma_K(Du(x)) < 1\}.$$

Proof. First suppose $u(x) = d_{K^\circ}(x) = \gamma_{K^\circ}(x - y)$ for some $y \in \partial U$. Then by Lemma 3 we have $D_l u(x) = -1$, where l is the direction of \overline{xy} segment (with $\gamma_{K^\circ}(l) = 1$). Therefore $\gamma_K(Du(x))$ can not be less than 1.

Next suppose $D_l u(x) = 1$ for some l with $\gamma_{K^\circ}(l) = 1$, and $u(x) < d_{K^\circ}(x)$. We know that Du is harmonic in the component of the open set $u < d_{K^\circ}$ containing x . As $|D_l u| \leq 1$ on this open set, the strong maximum principle implies that $D_l u$ is constant 1 there. Now consider the line containing x in the l direction and suppose it intersects the boundary of the open set in y . If $y \in \partial U$, then moving along the line we see that $u = d_{K^\circ}$,³ contradicting the choice of x . If $y \in U$, then we must have $u(y) = d_{K^\circ}(y)$. Also since $D_l u(y) = 1$, l must be the direction that connects y to (one of) its d_{K° -closest point on ∂U . Again we can see that along the line we have $u = d_{K^\circ}$, which is a contradiction. \square

Definition 3. The set

$$\{x \in U \mid u(x) = d_{K^\circ}(x)\} = \{x \in U \mid \gamma_K(Du(x)) = 1\}$$

is called the **plastic** region and is denoted by P . The set

$$\{x \in U \mid u(x) < d_{K^\circ}(x)\} = \{x \in U \mid \gamma_K(Du(x)) < 1\}$$

is called the **elastic** region and is denoted by E .

Lemma 3. *Suppose $x \in P$, and $y \in \partial U$ satisfies $u(x) = \gamma_{K^\circ}(x - y)$. Then the segment \overline{xy} is entirely plastic.*⁴

Proof. Consider $v = u - d_{K^\circ}$. Then along the segment \overline{xy} we have $D_l d_{K^\circ} = -1$, where l is the direction of the segment with $\gamma_{K^\circ}(l) = 1$. Thus $D_l v \geq 0$ along the segment, as $|D_l u| \leq 1$. Now since $v(x) = v(y) = 0$, we have $v \equiv 0$. Therefore $u = d_{K^\circ}$ along the segment as desired. \square

Theorem 1. *We have*

$$R_{K^\circ,0} \subset E.$$

Proof. Suppose to the contrary that $x \in R_{K^\circ,0} \cap P$. Then there are at least two points $y, z \in \partial U$ such that

$$d_{K^\circ}(x) = \gamma_{K^\circ}(x - y) = \gamma_{K^\circ}(x - z).$$

Now by the above lemma we must have $\overline{xy}, \overline{xz} \subset P$. In other words $u = d_{K^\circ}$ on both \overline{xy} and \overline{xz} . This implies that

$$Du(x) \cdot \frac{y - x}{\gamma_{K^\circ}(y - x)} = -1 = Du(x) \cdot \frac{z - x}{\gamma_{K^\circ}(z - x)},$$

which is a contradiction. \square

³Actually, along this segment we have $u(\cdot) = d_{K^\circ}(\cdot, y) \geq d_{K^\circ}(\cdot)$.

⁴This segment is obviously inside U .

3. THE CASE OF p -NORMS

From now on we consider γ_{K° to be the p -norm, γ_p , with $p \geq 2$, where

$$\gamma_p((x_1, x_2)) := (|x_1|^p + |x_2|^p)^{\frac{1}{p}}.$$

We denote the corresponding ridge by R_p and call it the p -ridge. In this case, the minimizer is in $C_{\text{loc}}^{1,1}(U)$ (see Safdari [12]).

Lemma 4. *The $R_{p,0}$ of a disk, i.e. the set of points inside the disk which have more than one p -closest point on its boundary, consists of the union of its diagonals parallel to the coordinate axes.*

Proof. It is easy to see that we can inscribe a p -circle inside the disk centered on the prescribed diagonals and touching the boundary of the disk in at least two points, so the $R_{p,0}$ contains them. Using Lagrange multipliers we can see that centered at any other point in the disk, we can inscribe a p -circle touching its boundary in exactly one point. \square

Next let us take the circle $(x - a)^2 + (y - b)^2 = r^2$ and assume that at the point (x_0, y_0) on it, the inscribed p -circle $|x - c|^p + |y - d|^p = s^p$ is tangent to the circle. We are interested in the direction $(c - x_0, d - y_0)$. This is the direction of the segment starting at (x_0, y_0) along which, (x_0, y_0) is the p -closest point on the circle to its points (for nearby points).

At (x_0, y_0) the tangents, hence the normals, of the circle and the p -circle are parallel. Therefore we have

$$2(a - x_0, b - y_0) = \lambda p((c - x_0)|c - x_0|^{p-2}, (d - y_0)|d - y_0|^{p-2}).$$

Thus the direction of $(c - x_0, d - y_0)$ is (parallel to)

$$(3.1) \quad (f_p(a - x_0), f_p(b - y_0)),$$

where f_p is the inverse of the one to one map $t \mapsto t|t|^{p-2} = \text{sgn}(t)|t|^{p-1}$. It is easy to see that f_p is odd, and $|f_p(t)| = |t|^{\frac{1}{p-1}}$. Also for nonzero t we have $f_p(t) = t|t|^{\frac{2-p}{p-1}}$, and $f'_p(t) = \frac{1}{p-1}|t|^{\frac{2-p}{p-1}}$. The following lemma is easy to prove.

Lemma 5. *If $x_0 \neq a$, $y_0 \neq b$, then there is an inscribed p -circle inside the circle which is touching the circle only at (x_0, y_0) .*

Next, we introduce a notion of curvature for curves in the plane.

Definition 4. The p -curvature of a curve $(x(t), y(t))$ in the plane is

$$(3.2) \quad \kappa_p := \frac{x'y'' - y'x''}{(p-1)|x'|^{\frac{p-2}{p-1}}|y'|^{\frac{p-2}{p-1}}(|x'|^{\frac{p}{p-1}} + |y'|^{\frac{p}{p-1}})^{\frac{p+1}{p}}}.$$

It is easy to see that κ_p does not change under the reparametrization of the curve, hence it is an intrinsic quantity.

Note that the p -curvature is not defined at the points where the tangent (hence the normal) to the curve is parallel to the coordinate axes. It also has the same sign as the ordinary curvature. It is also easy to see that the p -curvature of a p -circle is the inverse of its p -radius at all points where it is defined.

One notable difference between the usual curvature and the p -curvature for $p > 2$ is that κ_p does not appear in the first variation of the p -length.

Lemma 6. Suppose $x_i \in U$ converge to $x \in U$, and $y \in \partial U$ is the unique p -closest point to x . If $y_i \in \partial U$ is a (not necessarily unique) p -closest point to x_i , then y_i converges to y .⁵

Proof. Suppose that this does not happen. Then a subsequence of y_i , which we denote it again by y_i , will remain outside an open ball around y . Take this ball to be small enough so that x lies outside of it. Now consider a p -circle around x that touches ∂U only at y (The p -radius of the p -circle is obviously $d_p(x)$). Consider the subsets of this p -circle and ∂U that lie outside the above ball. These subsets are compact sets that do not intersect. Therefore they have a positive p -distance, δ , of each other. Let $\epsilon < \delta/2$ be small enough so that the p -circle of p -radius ϵ around x is inside the above p -circle, and does not touch the ball around y . As x_i approach x , they will be inside the p -circle with p -radius ϵ . Thus

$$d_p(x) + \delta \leq \gamma_p(x - y_i) \leq \gamma_p(x - x_i) + \gamma_p(x_i - y_i) < \epsilon + \gamma_p(x_i - y_i).$$

Hence

$$d_p(x) + \delta - \epsilon < d_p(x_i).$$

But, as d_p is continuous, for small enough ϵ we must have

$$d_p(x_i) < d_p(x) + \delta/2,$$

which is a contradiction. \square

We now impose some extra restrictions on ∂U , in order to study the regularity of d_p . We denote the inward normal to ∂U by ν .

Assumption 1. We assume that at the points where the normal to ∂U is parallel to one of the coordinate axes, the curvature of ∂U is small. In the sense that, if we have $(t, b(t))$ as a nondegenerate $C^{m,\alpha}$ ($m \geq 2$, $0 \leq \alpha < 1$) parametrization of ∂U around $y_0 := (0, b(0))$, and $b'(0) = 0$; then we assume b' goes fast enough to 0 so that $b'(t) = c(t)|c(t)|^{p-2}$, where $c(0) = 0$, and c is $C^{m-1,\alpha}$.

Also we require $c'(0)$ to be less than 1, and small enough so that $1 - c'(0)d_p(\cdot)$ does not vanish at the points inside U that have y_0 as the only p -closest point on ∂U .⁶

We will call these points the degenerate points of Assumption 1.

Remark 1. These assumptions will imply that there is a p -circle inside U touching ∂U only at y_0 .

Theorem 2. Suppose ∂U is $C^{m,\alpha}$ ($m \geq 2$, $0 \leq \alpha < 1$) or analytic, and satisfies Assumption 1. Then outside $R_{p,0}$, $d_p = d_p(\cdot, \partial U)$ is $C^{m,\alpha}$ or analytic, except at the points with

$$(3.3) \quad \kappa_p(y(x))d_p(x) = 1.$$

Where $y(x)$ is the p -closest point on ∂U to x .

Proof. Suppose $y \in \partial U$ and $\nu(y) = (a(y), b(y))$ is not parallel to the coordinate axes. Then as ν is smooth, it is not parallel to the coordinate axes in a neighborhood of y . Also as ∂U is smooth, there is a circle inside U that is tangent to ∂U only at y . Then $\nu(y)$ is also normal to the circle at y . By Lemma 5 there is a p -circle inside the circle and tangent to it (only) at y . Then we know that the points on the segment joining y to the center of the p -circle have y as the p -closest point to them on ∂U . Also

$$(3.4) \quad \mu(y) = \frac{1}{(|a|^{\frac{p}{p-1}} + |b|^{\frac{p}{p-1}})^{\frac{1}{p}}}(f_p(a), f_p(b))$$

⁵This lemma does not need any assumptions about ∂U .

⁶The requirement that $c'(0) < 1$ is only used to show that there is a p -circle in U that touches ∂U only at y_0 , but after examining this more carefully, it became apparent that this restriction is superfluous.

is the direction along which y is the p -closest point. Note that by the above assumption on the ∂U , this formula also gives the correct direction at points where one of a or b is zero. Also note that as $f_p(t)$ has the same sign as t , μ is pointing inward U .

Definition 5. We call μ the **inward p -normal** to ∂U at y .

If two points have the same point on ∂U as their p -closest point, then a p -circle centered at them will be tangent to ∂U at that point. As the inward normal to a point on a p -circle uniquely determines the point, the two points must be collinear with the point on the boundary. Hence the above normal bundle μ to ∂U gives at each point the only direction along which the point is the p -closest one.

Let x_0 be a point in $U - R_{p,0}$ and $y_0 = y(x_0)$. Let $y(t) = (a(t), b(t))$ for $|t| < L$, be a nondegenerate $C^{m,\alpha}$ or analytic parametrization of a segment of ∂U with $y(0) = y_0$. Also suppose that $(a + \epsilon, b)$ is in U for small positive ϵ . Now consider the map

$$F : (t, d) \mapsto y(t) + \mu(y(t))d$$

from the open set $(-L, L) \times (0, \infty)$ into \mathbb{R}^2 . We have $F(0, d_p(x_0)) = x_0$. We wish to compute DF around this point. First we deal with case that none of the a', b' vanish at 0 (and hence around 0). We have $\nu(t) = (-b'(t), a'(t))$ and

$$(3.5) \quad \mu(t) = \frac{(-f_p(b'(t)), f_p(a'(t)))}{(|a'(t)|^{\frac{p}{p-1}} + |b'(t)|^{\frac{p}{p-1}})^{\frac{1}{p}}}.$$

So we have

$$(3.6) \quad \begin{aligned} \mu'(t) &= \frac{(-(a')^2 b'' + a' b' a'', (b')^2 a'' - a' b' b'')}{(p-1)|a'|^{\frac{p-2}{p-1}}|b'|^{\frac{p-2}{p-1}}(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{p+1}{p}}} \\ &= -\kappa_p(a', b'). \end{aligned}$$

Hence

$$(3.7) \quad DF = \begin{pmatrix} a' - \kappa_p a' d & b' - \kappa_p b' d \\ \frac{-b'|b'|^{\frac{2-p}{p-1}}}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{1}{p}}} & \frac{a'|a'|^{\frac{2-p}{p-1}}}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{1}{p}}} \end{pmatrix}.$$

Thus

$$(3.8) \quad \det DF = (|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{p-1}{p}} (1 - \kappa_p d) = \gamma_q((a', b'))(1 - \kappa_p d),$$

where q is the dual exponent to p . This implies that F is $C^{m-1,\alpha}$ around $(0, d_p(x_0))$ with a $C^{m-1,\alpha}$ inverse, since $\kappa_p(y_0)d_p(x_0) \neq 1$.

Next suppose one of the a', b' vanishes. Thus the normal to ∂U is parallel to the coordinate axes. We assume that $b'(0) = 0$. The other case is similar. Note that by the assumption on ∂U we still have

$$\mu(t) = \frac{(-f_p(b'(t)), f_p(a'(t)))}{(|a'(t)|^{\frac{p}{p-1}} + |b'(t)|^{\frac{p}{p-1}})^{\frac{1}{p}}}.$$

The problem is that f_p is not differentiable at 0. To avoid this problem, we use the assumption that b' goes fast enough to 0. In other words, $b'(t) = c(t)|c(t)|^{p-2}$ where $c(0) = 0$, and c is $C^{m-1,\alpha}$. Hence $f_p(b'(t)) = c(t)$. We can also assume that $a(t) = t$ with no loss of generality, due to the inverse function theorem. Therefore

$$(3.9) \quad \mu(t) = \frac{(-c, 1)}{(1 + |c|^p)^{\frac{1}{p}}},$$

and

$$\begin{aligned}
\mu'(t) &= \frac{(-c', 0)}{(1 + |c|^p)^{\frac{1}{p}}} - \frac{c|c|^{p-2}c'(-c, 1)}{(1 + |c|^p)^{\frac{p+1}{p}}} \\
(3.10) \quad &= \frac{-c'}{(1 + |c|^p)^{\frac{p+1}{p}}}(a', b').
\end{aligned}$$

(Note that at $t \neq 0$ we have $\kappa_p = \frac{c'}{(1 + |c|^p)^{\frac{p+1}{p}}}$,⁷ and this expression is continuous at 0 with the value $c'(0)$ there.) Hence

$$(3.11) \quad DF = \begin{pmatrix} 1 - \frac{c'}{(1 + |c|^p)^{\frac{p+1}{p}}}d & c|c|^{p-2} - \frac{c'c|c|^{p-2}}{(1 + |c|^p)^{\frac{p+1}{p}}}d \\ \frac{-c}{(1 + |c|^p)^{\frac{1}{p}}} & \frac{1}{(1 + |c|^p)^{\frac{1}{p}}} \end{pmatrix}.$$

Thus

$$(3.12) \quad \det DF = (1 + |c|^p)^{\frac{p-1}{p}} \left(1 - \frac{c'}{(1 + |c|^p)^{\frac{p+1}{p}}}d\right).$$

Since we are only interested at $t = 0$,

$$(3.13) \quad \det DF(0, d) = 1 - c'(0)d.$$

We assumed that $c'(0)$ is small enough so that inside U this determinant does not vanish. Therefore F is $C^{m-1, \alpha}$ at these points too, with a $C^{m-1, \alpha}$ inverse around them.

Since $F : (t, d) \mapsto x$ is invertible in a neighborhood of $(0, d_p(x_0))$, we have

$$(3.14) \quad x = F(t(x), d(x)) = y(t(x)) + \mu(t(x))d(x).$$

We also know that in general

$$x = y(x) + \mu(y(x))d_p(x).$$

Note that $y(x)$ need not be unique for this formula to hold. If we take x close enough to x_0 , then by continuity $y(x), d_p(x)$ will be close to $y(x_0), d_p(x_0)$. And by invertibility of F we get

$$y(x) = y(t(x)), \quad d_p(x) = d(x).$$

As we showed that $x \mapsto (t, d)$ is locally $C^{m-1, \alpha}$, we obtain that $d_p(x)$ and $y(x)$ are also locally $C^{m-1, \alpha}$. Note that this also shows that all points around x_0 have a unique p -closest point around y_0 , which by continuity is the unique p -closest point to them on ∂U .

Next let us compute the first derivative of d_p and t . Looking at DF we can compute

$$(3.15) \quad DF^{-1} = \begin{pmatrix} \frac{a'|a'|^{\frac{2-p}{p-1}}}{(1 - \kappa_p d)(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})} & \frac{-b'}{\gamma_q(a', b')} \\ \frac{b'|b'|^{\frac{2-p}{p-1}}}{(1 - \kappa_p d)(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})} & \frac{a'}{\gamma_q(a', b')} \end{pmatrix}.$$

Which implies

$$(3.16) \quad Dd_p(x) = \left(\frac{-b'}{\gamma_q((a', b'))}, \frac{a'}{\gamma_q((a', b'))} \right),$$

⁷This equality only holds at the points where $c \neq 0$. But if we define κ_p to be c' at the points where $c = 0$, κ_p becomes continuous.

and

$$(3.17) \quad Dt(x) = \frac{(a'|a'|^{\frac{2-p}{p-1}}, b'|b'|^{\frac{2-p}{p-1}})}{(1 - \kappa_p d)(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})}.$$

Specially note that as a', b', t are $C^{m-1, \alpha}$, d_p is $C^{m, \alpha}$. A similar conclusion can be made for the degenerate points noting that

$$(3.18) \quad DF^{-1} = \begin{pmatrix} \frac{1}{(1+|c|^p)(1 - \frac{c'}{(1+|c|^p)^{\frac{p+1}{p}}}d)} & \frac{-c|c|^{p-2}}{(1+|c|^p)^{\frac{p-1}{p}}} \\ \frac{c}{(1+|c|^p)(1 - \frac{c'}{(1+|c|^p)^{\frac{p+1}{p}}}d)} & \frac{1}{(1+|c|^p)^{\frac{p-1}{p}}} \end{pmatrix}.$$

□

Theorem 3. *We have*⁸

$$(3.19) \quad \Delta d_p = \frac{-(p-1)\tau_p \kappa_p}{1 - \kappa_p d_p},$$

where

$$(3.20) \quad \tau_p = \frac{(|b'|^{\frac{2}{p-1}} + |a'|^{\frac{2}{p-1}})|a'b'|^{\frac{p-2}{p-1}}}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{2p-2}{p}}}$$

is a positive reparametrization invariant quantity along ∂U . At the degenerate points of Assumption 1 we have $\Delta d_p = 0$.

Proof. We know that

$$Dd_p(x) = \left(\frac{-b'}{\gamma_q(a', b')}, \frac{a'}{\gamma_q(a', b')} \right) = \frac{(-b'(t), a'(t))}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{p-1}{p}}}.$$

Therefore

$$\begin{aligned} D_{11}d_p &= \\ &= \left[\frac{-b''}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{p-1}{p}}} + \frac{b'(a'|a'|^{\frac{2-p}{p-1}}a'' + b'|b'|^{\frac{2-p}{p-1}}b'')}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{2p-1}{p}}} \right] D_1 t \\ &= \frac{[-|a'|^{\frac{p}{p-1}}b'' + b'a'a'|^{\frac{2-p}{p-1}}a'']a'|^{\frac{2-p}{p-1}}}{(1 - \kappa_p d)(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{3p-1}{p}}}. \end{aligned}$$

Similarly

$$\begin{aligned} D_{22}d_p &= \\ &= \left[\frac{a''}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{p-1}{p}}} - \frac{a'(a'|a'|^{\frac{2-p}{p-1}}a'' + b'|b'|^{\frac{2-p}{p-1}}b'')}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{2p-1}{p}}} \right] D_2 t \\ &= \frac{[|b'|^{\frac{p}{p-1}}a'' - b'a'|^{\frac{2-p}{p-1}}b'']b'|^{\frac{2-p}{p-1}}}{(1 - \kappa_p d)(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{3p-1}{p}}}. \end{aligned}$$

⁸Outside R_p .

Hence

$$\begin{aligned}
\Delta d_p &= \\
&= \frac{a''b'(|b'|^{\frac{2}{p-1}} + |a'|^{\frac{2}{p-1}}) - b''a'(|b'|^{\frac{2}{p-1}} + |a'|^{\frac{2}{p-1}})}{(1 - \kappa_p d)(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{3p-1}{p}}} \\
&= \frac{-(p-1)\kappa_p}{1 - \kappa_p d} \frac{(|b'|^{\frac{2}{p-1}} + |a'|^{\frac{2}{p-1}})|a'b'|^{\frac{p-2}{p-1}}}{(|a'|^{\frac{p}{p-1}} + |b'|^{\frac{p}{p-1}})^{\frac{2p-2}{p}}} \\
&= \frac{-(p-1)\tau_p \kappa_p}{1 - \kappa_p d_p}.
\end{aligned}$$

For the degenerate case, we just need to note that by our assumptions Δd_p is continuous. Thus if we rewrite the above formula in this case (for $t \neq 0$) we get

$$(3.21) \quad \Delta d_p = \frac{-(p-1)\tau_p \kappa_p}{1 - \kappa_p d} = \frac{-(p-1)(1+c^2)|c|^{p-2}c'}{(1+|c|^p)^{\frac{3p-1}{p}}} \frac{1}{1 - \frac{c'd_p}{(1+|c|^p)^{\frac{p+1}{p}}}}.$$

And as $t \rightarrow 0$ this expression goes to 0. Hence $\Delta d_p = 0$ in this case. \square

Theorem 4. *The p -ridge consists of $R_{p,0}$ and those points outside of it at which*

$$1 - \kappa_p d_p = 0.$$

Proof. First note that we assumed as before that $1 - c'(0)d_p$ does not vanish inside U .⁹ Thus we do not need to consider the degenerate case.

So far we showed that the p -ridge contains $R_{p,0}$, and every other point at which $1 - \kappa_p d_p \neq 0$ is not in the p -ridge. Now suppose $1 - \kappa_p(y(x))d_p(x) = 0$. Then $\kappa_p(y) = \frac{1}{d_p(x)} > 0$. We claim that Δd_p blows up as we approach x from a neighborhood of the line segment \overline{xy} . The reason is that on this segment $1 - \kappa_p d_p \neq 0$ (hence also in a neighborhood of it by continuity). Thus we can apply the above theorems to show that d_p is at least C^2 on a neighborhood of the segment, and its Laplacian is given by the formula (3.19). Therefore d_p blows up, and can not be $C^{1,1}$ in any neighborhood of x . \square

Theorem 5. *The p -ridge is elastic.*

Proof. We already showed that $R_{p,0}$ is elastic. Therefore consider a (nondegenerate) point x not in $R_{p,0}$ at which $1 - \kappa_p(y(x))d_p(x) = 0$, and suppose it is plastic. (Note that $\kappa_p(y(x)) > 0$.) Then we know that the segment between x and $y(x)$ is also plastic. Hence along the segment $u = d_p$. As $u \leq d_p$ in general, for a in the segment we have

$$(3.22) \quad \Delta_{h,\xi}^2 u(a) := \frac{1}{h^2}(u(a-h\xi) + u(a+h\xi) - 2u(a)) \leq \Delta_{h,\xi}^2 d_p(a).$$

Where ξ is the direction orthogonal to the direction of the segment, $\zeta := \frac{x-y}{|x-y|}$. Note that d_p is linear along the segment so $D_{\zeta\zeta}^2 d_p$ vanishes there. Hence $\Delta d_p = D_{\xi\xi}^2 d_p$ on the segment. Therefore

$$\lim_{a \rightarrow x} D_{\xi\xi}^2 d_p(a) = \lim_{a \rightarrow x} \frac{-(p-1)\tau_p(y)\kappa_p(y)}{1 - \kappa_p(y)d_p(a)} = -\infty.$$

Thus for large positive M we can choose a close to x , and $h(a)$ small enough, such that for $h \leq h(a)$

$$\Delta_{h,\xi}^2 u(a) \leq \Delta_{h,\xi}^2 d_p(a) < -M.$$

⁹If we do not assume this, we cannot carry over the proof of this theorem to the case of degenerate points of Assumption 1, since $\Delta d_p = 0$ there.

Fixing a and taking the limit as $h \rightarrow 0$ we get

$$D_{\xi\xi}^2 u(a) \leq -M.$$

(D^2u must actually be interpreted as the Lipschitz constant of Du .)¹⁰ But this contradicts $C^{1,1}$ regularity of u around x . \square

Lemma 7. $1 - \kappa_p d_p > 0$ on the free boundary.

Proof. Suppose x_0 is on the free boundary, and y_0 is the p -closest point on ∂U to it. As the free boundary is part of the plastic region, $1 - \kappa_p(y_0)d_p(x_0)$ is nonzero. But if it was negative we would have

$$1 - \kappa_p(y_0)d_p(x') = 0$$

for some x' in the segment $\overline{x_0 y_0}$. Since d_p linearly grows along this segment and it is zero at y_0 . Thus x' must be in the elastic region contradicting the fact that the segment is plastic.

In the degenerate case of Assumption 1 we need to replace $1 - \kappa_p d_p$ with $\det DF$. The proof is the same. \square

4. DOMAINS WITH CORNERS

We now relax some of the constraints on ∂U . We assume that ∂U consists of a finite number of (at least) C^2 (up to the endpoints) disjoint arcs S_1, \dots, S_m . We denote by V_i the vertex $S_i \cap S_{i+1}$, and by α_i the angle formed by S_i, S_{i+1} .¹¹

Definition 6. We say the vertex V_i is a **reentrant corner** if $\alpha_i \geq \pi$, a **strict reentrant corner** if $\alpha_i > \pi$, and a **nonreentrant corner** if $\alpha_i < \pi$.¹²

Let us look more closely at the arguments in the proof of regularity of d_p . We see that as long as the point $y_0 \in \partial U$ is not the p -closest point on ∂U to any point in $U \setminus R_{p,0}$, we do not use any regularity assumption about ∂U at y_0 . Therefore instead of Assumption 1 we can assume

Assumption 2. If $y_0 \in \partial U$ is a point where the normal to the boundary at it is parallel to one of the coordinate axes, then there is no p -circle inside U that touches its boundary only at y_0 .

In this case the p -ridge can extend to ∂U at y_0 in the direction of the normal to it. But it will still remain inside the elastic region. (Note that by definition, the p -ridge is closed relative to U . Thus it remains outside a U -neighborhood of the free boundary.)

A sufficient condition for this assumption to hold, is that we can parametrize ∂U around these points as $(t, b(t))$ where $b'(0) = 0$ and $b''(0) > 0$. This is true, for example, when U is a disk.

It is also easy to see that as p -circles have tangent lines at every point, nonreentrant corners can not be the p -closest point on ∂U to any point in U .

We have the following generalization of Theorem 2.

¹⁰In fact $\Delta_{h,\xi}^2 u(a) = \frac{1}{h}(D_\xi u(a + h_1 \xi) - D_\xi u(a - h_2 \xi))$ for some $0 \leq h_i \leq h$. Thus we must have $|\Delta_{h,\xi}^2 u(a)| \leq 2\|Du\|_{C^{0,1}}$.

¹¹We still assume that each component of U is the interior of a simple closed Jordan curve with finitely many holes.

¹²When one of the normals to ∂U at a nonreentrant corner is parallel to one of the coordinate axes, we do not need to assume Assumptions 1 or 2 hold there, except in Section 7.

When V_i is a reentrant corner and the normal to S_i at it is parallel to coordinate axes, either Assumption 1 must hold for S_i at its endpoint V_i , or the modified version of Assumption 2 must hold, i.e. we assume that there is no p -circle inside U whose center is on the p -normal to S_i at V_i , and touches ∂U only at V_i .

Theorem 6. Suppose ∂U is piecewise $C^{m,\alpha}$ for $(m \geq 2, 0 \leq \alpha < 1)$ or piecewise analytic, and satisfies Assumptions 1 or 2. Then outside $R_{p,0}$, $d_p = d_p(\cdot, \partial U)$ is $C^{m,\alpha}$ or analytic around the points with $\kappa_p(y(x))d_p(x) \neq 1$, where $y(x)$ is the p -closest point on ∂U to x and is not a reentrant corner.

If $y(x)$ is a reentrant corner and the segment \overline{xy} is between the inward p -normals to y , then d_p is analytic around x . And if the segment \overline{xy} coincides with one of the inward p -normals (or both of them if they coincide) and $\kappa_p(y(x))d_p(x) \neq 1$, where κ_p is the p -curvature of the corresponding boundary part, then d_p is $C^{1,1}$ around x . (But not C^2 in general.)

Proof. We just need to consider reentrant corners. Suppose $y_0 = V_1$ is a strict reentrant corner and it is the p -closest point to x_0 . Let μ_1, μ_2 be the inward p -normals to respectively S_1, S_2 at y_0 . We assumed that if any of the μ_i 's is parallel to coordinate axes then one of the Assumptions 1 or 2 holds.

Let l be the tangent line at y_0 to C_0 , where C_0 is the p -circle around x_0 which passes through y_0 . l must be between l_1, l_2 , the tangent lines to S_1, S_2 at y_0 . Because the p -circle is inside the domain. If $l \neq l_1, l_2$ then we claim that

$$d_p(x) = \gamma_p(x - y_0)$$

for x close to x_0 . Therefore d_p is analytic there. To prove the claim consider C , the p -circle around x that passes through y_0 . Then C is in a small neighborhood of C_0 . As C_0 is inside U except at y_0 and U is open, we can see that C is inside U except possibly at a neighborhood of y_0 . But the tangent to C at y_0 is close to l thus it is also strictly between l_1, l_2 . Therefore C is inside U and touches ∂U only at y_0 .

Next we consider the case that $l = l_1$. Then x_0 is in the direction of μ_1 . If μ_1 is parallel to one of the coordinate axes, Assumption 1 must hold. We assume that $1 - \kappa_p(y_0)d_p(x_0) \neq 0$ where $\kappa_p(y_0)$ is the p -curvature of S_1 at y_0 . Now let us extend S_1 in a smooth way so that C_0 still stays inside the new domain, and the new boundary stays below l outside of a small neighborhood of x_0 . Then d_p does not change around x_0 on the side of μ_1 that S_1 is. The reason is that the p -circles around those points which touch S_1 are either completely on the mentioned side of μ_1 , or intersect the interior of the segment $\overline{x_0 y_0}$. The later p -circles intersect C_0 close to x_0 on the side of S_1 , since they also touch S_1 outside C_0 and close to x_0 . Now note that two p -circles whose centers and p -radii are close to each other will intersect at exactly two almost antipodal points. Therefore the mentioned p -circles can not intersect the new boundary part, which lies outside of C_0 on the opposite side of S_1 . Here we used the fact that two distinct p -circle can intersect at at most two points. To see this, note that two p -circles cannot intersect more than once on one side of the line that joins their centers. Otherwise we can shrink or expand one of them so that they become tangent at one point on that side of the line. But in this situation the centers and the point of tangency must be collinear, which is impossible.

Hence we can apply the regularity result for d_p in the case of smooth boundaries. Now consider a small disk around x_0 and divide it by μ_1 . Then d_p is smooth up to the boundary of the half disk on the same side as S_1 . On the other half, d_p is the p -distance from y_0 hence it is analytic. Let us compute $Dd_p(x_0)$ from both sides. On the side of S_1 we have

$$Dd_p(x_0) = \left(\frac{-b'}{\gamma_q((a', b'))}, \frac{a'}{\gamma_q((a', b'))} \right) = \frac{\nu(y_0)}{\gamma_q(\nu(y_0))},$$

where $\nu(y_0)$ is the inward normal to S_1 at y_0 (note that this formula is also true in the degenerate case of Assumption 1). On the other side we have

$$Dd_p(x_0) = D_i \gamma_p(x_0 - y_0) = \frac{((x_0)_i - (y_0)_i) |(x_0)_i - (y_0)_i|^{p-2}}{\gamma_p(x_0 - y_0)^{p-1}}.$$

But we have $\mu_1 = \frac{x_0 - y_0}{\gamma_p(x_0 - y_0)}$, thus

$$(4.1) \quad D_i d_p(x_0) = \mu_{1i} |\mu_{1i}|^{p-2}.$$

On the other hand

$$\mu_{1i} = \frac{f_p(\nu_i(y_0))}{\gamma_q(v(y_0))^{\frac{1}{p-1}}},$$

where f_p is the inverse of $t \mapsto t|t|^{p-2}$ as before. Therefore the derivative of d_p is the same from both sides and it is C^1 around x_0 . As d_p is smooth on both sides of μ_1 up to μ_1 , we can say that it is in fact $C^{1,1}$ around x_0 . But it is not in general C^2 there. To see this we compute Δd_p from both sides. On the side of S_1 we have

$$\Delta d_p(x_0) = \frac{-(p-1)\tau_p(y_0)\kappa_p(y_0)}{1 - \kappa_p(y_0)d_p(x_0)}.$$

And on the other side

$$(4.2) \quad \begin{aligned} \Delta d_p(x_0) &= \sum D_{ii}^2 \gamma_p(x_0 - y_0) \\ &= (p-1) \sum \frac{|(x_0)_i - (y_0)_i|^{p-2}}{\gamma_p(x_0 - y_0)^{p-1}} - \frac{|(x_0)_i - (y_0)_i|^{2p-2}}{\gamma_p(x_0 - y_0)^{2p-1}}. \end{aligned}$$

Now if, for example, κ_p vanishes at y_0 and $(x_0)_i \neq (y_0)_i$ the two values will be different.

If $\alpha_1 = \pi$ (which means ∂U is C^1 at y_0), then we can repeat the same arguments. We can show that if a point x_0 has y_0 as the only p -closest point on the boundary and satisfies $1 - d_p(x_0)(\kappa_p)_i(y_0) \neq 0$, where $(\kappa_p)_i$ is the p -curvature of S_i (they can be different at y_0 as ∂U is not necessarily C^2 there), then d_p is $C^{1,1}$ in a neighborhood of x_0 . But it is also not in general C^2 . \square

We can also show that if $1 - \kappa_p(y_0)d_p(x_0) = 0$ then x_0 is in the ridge. The proof of this fact goes exactly as in the case of smooth boundaries. The only modification is that we may need to approach x_0 only from one side of the segment $\overline{x_0 y_0}$.

Theorem 7. *Suppose ∂U is piecewise $C^{m,\alpha}$ for $(m \geq 2, 0 \leq \alpha < 1)$ or piecewise analytic, and satisfies Assumptions 1 or 2. Then the p -ridge consists of $R_{p,0}$ and those points outside of it where $1 - \kappa_p d_p = 0$. (The necessary adjustment must be made in the case of reentrant corners.)¹³*

Also it should be noted that at points where d_p is smooth, we can still compute Δd_p as in Theorem 3 or formula (4.2).

5. PROOF OF THE REGULARITY

In this section we are going to modify the proof presented in Friedman [8] in order to be applicable to our situation. The main difficulty for doing so is that we are dealing with the inward p -normal instead of the inward normal.

First let us assume that ∂U is $C^{m,\alpha}$ for $(m \geq 3, 0 < \alpha < 1)$ or analytic, and satisfies Assumption 1 at degenerate points. We parametrize it by $y(t)$ for $0 \leq t \leq L$. As before we denote the inward p -normal by $\mu = \mu(y(t))$.

¹³The p -ridge is also elastic in this case. Although the proof of Theorem 5 must be modified to show that if x_0 belongs to one of the p -normals at a reentrant corner y_0 and $1 - \kappa_p(y_0)d_p(x_0) = 0$, then x_0 is elastic. In this case we need to use the fact that $Du = Dd_p$ on the interior of the segment $\overline{x_0 y_0}$ when x_0 is plastic. Then moving in the direction ξ orthogonal to the segment, we must have $D_\xi u \leq D_\xi d_p$ for some points arbitrarily close to the segment. This easily gives a contradiction with the boundedness of the $C^{0,1}$ norm of Du around x_0 .

Theorem 8. *There exists a nonnegative function $\delta(t)$, such that the plastic set is*

$$(5.1) \quad P = \{x \mid x = y(t) + s\mu(t), 0 \leq s \leq \delta(t), 0 \leq t \leq L\}.$$

Moreover we have $d_p(y(t) + s\mu(t)) = s$ for $0 \leq s \leq \delta(t)$.

Proof. Remember that for $x \in U$, $y(x)$ is the p -closest point on ∂U to x . We proved that if $x \in P$ then the segment $\overline{xy(x)}$ is also in P .

Consider the (connected) segment starting at the boundary point y in the direction of the inward p -normal at it, that lies completely in U . Along this segment we can look at the supremum of points that have y as the only p -closest point to them on the boundary. This supremum can not be on ∂U , as points close to that boundary point have p -closest point close to it.

Now consider the supremum of points on the segment that have y as the only p -closest point, and are in P . This supremum also belongs to P as P is closed. Also it has y as the only p -closest point on the boundary, as P does not intersect the p -ridge (specially $R_{p,0}$), and the p -closest point on the boundary changes continuously. In addition, this supremum belongs to U as it precedes the first supremum. \square

Remark 2. Note that

$$(5.2) \quad y(t) + (\delta(t) + \epsilon)\mu(t) \in E,$$

where $0 < \epsilon < \epsilon(t)$. The reason is that $x(t) = y(t) + \delta(t)\mu(t)$ is inside U by the above proof.¹⁴

If we show that δ is continuous, then Γ , the free boundary, is parametrized by $\delta(t)$ for $0 \leq t \leq L$.

Theorem 9. *δ is a continuous function of t .*

Proof. Suppose we want to show the continuity of δ at t_0 from right. Let $t_n \searrow t_0$ and suppose that $\delta(t_n) \rightarrow \delta(t_0) + \epsilon$ where $\epsilon > 0$. Then look at the points

$$x_n = y(t_n) + (\delta(t_0) + \epsilon/k)\mu(t_n),$$

where k is a big constant that makes $\tilde{x} = y(t_0) + (\delta(t_0) + \epsilon/k)\mu(t_0)$ remain inside the elastic region E . Now for large n , x_n is in the plastic region and we have $u(x_n) = d_p(x_n)$. But by continuity of u, d_p we must have $u(\tilde{x}) = d_p(\tilde{x})$, which is a contradiction. Therefore the limit points of the δ values of any sequence approaching t_0 must be in the interval $[0, \delta(t_0)]$. In particular if $\delta(t_0) = 0$ then δ is continuous at t_0 .

Now we need to show that the limit points can not be less than $\delta(t_0)$ either. First let us show that there is a sequence $t_n \searrow t_0$ such that $\delta(t_n) \rightarrow \delta(t_0)$. If this does not happen then $\delta(t_0)$ is outside the closed set of all limit points of δ -values of all sequences approaching t_0 from right. Therefore an interval of the form $[\delta(t_0) - \epsilon, \delta(t_0)]$ is out of this set. This means that for t close to t_0 we have $\delta(t) < \delta(t_0) - \epsilon$. So the open set

$$\{x \mid x = y(t) + s\mu(t), \delta(t_0) - \epsilon < s < \delta(t_0), t_0 < t < t + \epsilon'\}$$

is elastic and over it we have

$$-\Delta u = \eta.$$

(This set is an open set by invariance of domain, as it is the image of the locally injective continuous function $F(t, s) = y(t) + s\mu(t)$ introduced in the proof of Theorem 2. We just need to take ϵ, ϵ' small enough.) Thus u is analytic there. Now on the segment

$$\{y(t_0) + s\mu(t_0) \mid \delta(t_0) - \epsilon < s < \delta(t_0)\}$$

¹⁴When $\delta(t) = 0$, we use the fact that a one-sided neighborhood of ∂U is inside U .

we have $u = d_p$ and d_p is a linear function there. Also as $u - d_p$ achieves its maximum there and it is C^1 , we have $Du = Dd_p$. But Dd_p is constant along the segment and its value depends only on $\mu(t_0)$ there. Hence by uniqueness of the solution of the Cauchy problem, locally, u equals the linear function whose derivative is Dd_p plus a quadratic function of the euclidean distance from the line containing the segment. (Note that $-\Delta$ is elliptic so the segment is a noncharacteristic surface for it.) As these two functions are analytic, they must be equal on the component of the elastic region attached to the segment. But this component contains the set

$$\{y(t_0) + s\mu(t_0) \mid \delta(t_0) < s < \delta(t_0) + \tilde{\epsilon}\}.$$

Hence u equals the linear function there, as the quadratic part vanishes on it. However the linear function equals d_p along this segment too. Hence we get a contradiction.

Now suppose $s_n \searrow t_0$ and $\delta(s_n) \rightarrow \delta(t_0) - \epsilon$. By taking subsequences we can assume $t_{n+1} < s_n < t_n$ where $\delta(t_n) \rightarrow \delta(t_0)$. For n large enough, $\delta(s_n)$ is less than $\delta(t_n), \delta(t_{n+1})$. As we showed that δ is upper semicontinuous, it achieves its maximum over $[t_{n+1}, t_n]$. Let this maximum be M_n . The set $\{t \in (t_{n+1}, t_n) \mid \delta(t) < M_n\}$ must be open for the same reason. Let (t'_{n+1}, t'_n) be the component of it that contains s_n . Obviously the largest of $\delta(t'_{n+1}), \delta(t'_n)$ equals M_n . Now take the set

$$(5.3) \quad A_n = \{t'_{n+1} < t < t'_n, m_n < s < M_n\},$$

where m_n is the infimum of δ over $[t'_{n+1}, t'_n]$. We want $F(A_n)$ to be an open set. Set

$$x_0 = y(t_0) + \delta(t_0)\mu(t_0).$$

Lemma 7 implies

$$1 - \kappa_p(t_0)d_p(x_0) = 1 - \kappa_p(t_0)\delta(t_0) > 0.$$

(In the degenerate case of Assumption 1 we need to replace this expression with $\det DF$ at those points). We know that F is injective on a neighborhood of x_0 . We choose this neighborhood small enough so that $1 - \kappa_p d_p > 0$ over it too. Let n be large enough so that for $t_{n+1} < t < t_n$ the half-lines containing inward p -normals intersect this neighborhood. Then as $1 - \kappa_p d_p$ can change sign only once along these half-lines, we conclude that it is positive on $F(A_n)$ (Note that M_n is close to $\delta(t_0)$). Hence F is at least C^1 on A_n , and the determinant of its derivative never vanishes. This implies that F is injective on A_n . Otherwise the restriction of F to the segment connecting two points with the same F -value would have a local extremum, which implies DF has a zero eigenvalue there, contradicting the assumptions. Therefore by invariance of domain $F(A_n)$ is open.

Now consider the function $v := d_p - u$ over the open set

$$E_n := E \cap F(A_n).$$

We have $v > 0$ and $\Delta v = \Delta d_p + \eta$ over it. First let us show that Δv has a positive infimum over E_n . Lemma 8 (which does not assume any regularity about the free boundary) shows that $\Delta d_p + \eta > 0$ on the free boundary. Thus if we choose the neighborhood around x_0 small enough, then as the tale of each segment (along the p -normals) in $F(A_n)$ is in that neighborhood, $\Delta d_p + \eta$ will have a positive infimum there. Also it is positive at the free boundary point on each segment. We also know that

$$\Delta d_p + \eta = \eta - \frac{(p-1)\kappa_p \tau_p}{1 - \kappa_p d_p}.$$

Thus it can not change sign twice along these segments, as d_p grows linearly along them. (In the degenerate case of Assumption 1 we have $\Delta d_p = 0$ and this expression is certainly positive.) Hence $\Delta d_p + \eta > 0$ on E_n . And as it has a positive infimum on the tale of segments, and by compactness

a positive minimum on their initial points on the free boundary, we can say it has a positive infimum over E_n (note that it is monotone along segments). Therefore we have

$$\Delta v \geq \lambda > 0$$

on E_n for some constant λ .¹⁵ Now let

$$(5.4) \quad w(x) := v(x) - v(\bar{x}) - \frac{\lambda}{2}|x - \bar{x}|^2,$$

where

$$\bar{x} = y(s_n) + (\delta(s_n) + \epsilon')\mu(s_n) \in E_n.$$

Then we have

$$\Delta w = \Delta v - \lambda \geq 0,$$

and $w(\bar{x}) = 0$. So by the maximum principle $\sup w \geq 0$ and is attained on ∂E_n . But on the free boundary and on the segments

$$\{y(t) + s\mu(t) \mid t = t'_n, t'_{n+1}, m_n \leq s \leq \delta(t)\},$$

we have $v = 0$ so $w < 0$. Therefore $\sup w$ must be attained on

$$B_n := \{y(t) + M_n\mu(t) \mid t'_{n+1} \leq t \leq t'_n\} \cup \{y(t) + s\mu(t) \mid t = t'_n, t'_{n+1}, \delta(t) < s \leq M_n\}.$$

Hence for some x' in this set we have

$$v(x') - v(\bar{x}) - \frac{\lambda}{2}|x' - \bar{x}|^2 = w(x') \geq 0,$$

or (noting that $M_n \geq \delta(t'_n), \delta(t'_{n+1})$)

$$(5.5) \quad v(x') - v(\bar{x}) \geq \frac{\lambda}{2}|x' - \bar{x}|^2 \geq C \frac{\lambda}{2} \min\{|\delta(t'_{n+1}) - \delta(s_n) - \epsilon'|^2, |\delta(t'_n) - \delta(s_n) - \epsilon'|^2\},$$

where C depends on the maximum and minimum of the C^1 norm of F on a set of the form

$$\{t_0 \leq t \leq t_0 + \tilde{\epsilon}, 0 \leq s \leq M\},$$

which can be shown similar to before that F is C^1 over it with a C^1 inverse. Letting $\epsilon' \rightarrow 0$ we get

$$v(x') \geq C \frac{\lambda}{2} \min\{|\delta(t'_{n+1}) - \delta(s_n)|^2, |\delta(t'_n) - \delta(s_n)|^2\}.$$

However B_n intersects the free boundary at its end points. Let the closest endpoint to x' be x'' . Then as u is $C^{1,1}$ away from ∂U , we have

$$(5.6) \quad v(x') \leq C'|x'' - x'| + v(x'') \leq C'C[(t'_n - t'_{n+1}) + (M_n - \delta(t'_n)) + (M_n - \delta(t'_{n+1}))].$$

Where C' bounds $C^{1,1}$ norm of $v = d_p - u$ around x_0 , and C is as before. Therefore we have

$$(5.7) \quad \begin{aligned} |\delta(t_n) - \delta(s_n)|^2 &\leq \min\{|\delta(t'_{n+1}) - \delta(s_n)|^2, |\delta(t'_n) - \delta(s_n)|^2\} \\ &\leq \tilde{C}[(t'_n - t'_{n+1}) + (M_n - \delta(t'_n)) + (M_n - \delta(t'_{n+1}))] \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

But this contradicts the assumption. Hence δ is continuous at t_0 . □

¹⁵Note that λ is independent of n .

Remark 3. If we allow ∂U to be piecewise smooth and to satisfy Assumptions 1 or 2 at degenerate points, then we can still say that the free boundary is locally a continuous arc. To see this take x_0 to be a point on the free boundary (inside U), and let y_0 be the p -closest point to it on the boundary (we know that $x_0 \notin R_{p,0}$). If y_0 is a regular point or a degenerate point of Assumption 1, then around it we can still represent the free boundary as the graph of a function δ and the previous arguments apply. If y_0 is a reentrant corner and x_0 is between the inward p -normals at y_0 , then we can define a similar function δ (of the angle between $\overline{x_0 y_0}$ and one of the p -normals) whose graph is the free boundary and the analysis is similar to before. (Note that in this case $\Delta d_p \geq 0$ and d_p is just the p -distance from the corner, so the analysis is actually simpler.)

And finally if $\overline{x_0 y_0}$ is in the direction of one of the p -normals, then we can prove the continuity from each side separately. Although the above proof needs modifications as we did not show that $1 - \kappa_p d_p$ is necessarily nonzero at these points. Therefore the “open” set in the proof is not necessarily the image of an injective map.¹⁶ Nevertheless the analysis below does not apply at these points and we can not get regularity at them. So we do not go into the details of this case.

Now we can apply the following theorem proved in Friedman [8]. Note that the free boundary is locally a Jordan arc, since it is parametrized by the continuous injective map

$$t \mapsto y(t) + \delta(t)\mu(t).$$

Injectivity follows from the arguments in the beginning of this section resulted in the definition of δ , and the fact that the free boundary is part of the plastic region. (Thus points of the free boundary have a unique p -closest point on ∂U to them.) The case of reentrant corners is similar.

Theorem 10. *Let V be a simply connected domain in \mathbb{R}^2 whose boundary ∂V is a Jordan curve, and let $\Gamma \subset \partial V \cap B_R(x_0)$ be a Jordan arc (for some $R > 0$). Also assume that $u \in C^{1,1}(V \cup \Gamma)$, $f \in C^{m,\alpha}(B_R(x_0))$, $\phi \in C^{m+2,\alpha}(B_R(x_0))$ where $m \geq 0$, $0 < \alpha < 1$. And assume that u satisfies*

$$\begin{aligned} \Delta u &= f && \text{in } V \cap B_R(x_0) \\ (5.8) \quad u &= \phi, \quad Du = D\phi && \text{on } \Gamma \\ f - \Delta \phi &\neq 0 && \text{in } B_R(x_0). \end{aligned}$$

Then Γ has a $C^{m+1,\alpha}$ nondegenerate parametrization.

Now we can take x_0 on the free boundary (inside U) to be a point where d_p is at least $C^{3,\alpha}$ for $0 < \alpha < 1$ around it. Take V to be the subset of E , the elastic set, consisted of points above the graph of δ around x_0 . Note that V is simply connected as we can project any loop in it onto the graph of δ and then shrink it to a point. We know that u is in $C^{1,1}(\bar{V})$. If we show that $-\eta - \Delta d_p \neq 0$ at x_0 , then by continuity we can choose a small ball $B_R(x_0)$ such that $-\eta - \Delta d_p \neq 0$ everywhere on it, and the theorem applies. We use the following lemma from Friedman [8] to show this. Note that our

¹⁶The one-sided continuity from the region between the two p -normals follows as before, if we are not in the degenerate case of Assumption 2. Continuity from the other side follows similarly in some cases, for example when $\kappa_p = 0$ at the corner.

problem is equivalent to

$$\begin{aligned}
(5.9) \quad & -\Delta(-u) + \eta \geq 0 \\
& -u \geq -d_p \\
& (-\Delta(-u) + \eta)(-u + d_p) = 0.
\end{aligned}$$

Lemma 8. *Suppose*

$$\begin{aligned}
& -\Delta u + f \geq 0 \\
& u \geq \phi
\end{aligned}$$

$$(-\Delta u + f)(u - \phi) = 0$$

in U , and $u \in C^{1,1}(U)$, $\phi \in C^3$. Then on the free boundary, if $f - \Delta\phi$ and $D(f - \Delta\phi)$ do not vanish simultaneously, we have $f - \Delta\phi > 0$. (The free boundary is the boundary of the set $\{u > \phi\}$ inside U .)

Hence we only need to show that $\psi := \eta + \Delta d_p > 0$ along the free boundary, by showing that $\psi, D\psi$ can not vanish simultaneously there. In order to prove the latter fact, suppose at a nondegenerate point x_0 we have

$$\psi = \eta + \Delta d_p = \eta - \frac{(p-1)\kappa_p\tau_p}{1 - \kappa_p d_p} = 0.$$

As η, τ_p are positive, this implies that $\kappa_p(y(x_0)) \neq 0$. (Note that as free boundary is plastic we have $1 - \kappa_p d_p \neq 0$ there.) Now we have

$$(5.10) \quad D_\mu \psi = D_\mu \Delta d_p = \frac{-(p-1)\kappa_p^2 \tau_p}{(1 - \kappa_p d_p)^2} D_\mu d_p = \frac{-(p-1)\kappa_p^2 \tau_p}{(1 - \kappa_p d_p)^2} \neq 0.$$

Since κ_p, τ_p do not change along $\mu = \mu(y(x_0))$ and $D_\mu d_p = 1$. In case of degenerate points of Assumption 1, we showed that there $\Delta d_p = 0$ so $\psi > 0$. Finally if $y(x_0)$ is a reentrant corner and x_0 is between the inward p -normals, formula (4.2) shows that $\Delta d_p \geq 0$, hence $\psi > 0$.

Now if we show that the free boundary has no cusps, we have proved that it is a smooth curve. The following theorem proved in Friedman [8] for $p = 2$, but the proof works in this more general setting with no change.

Theorem 11. *The plastic set has positive density at each point of the free boundary where d_p is $C^{3,\alpha}$ around it, for some $0 < \alpha < 1$.*

Therefore putting all these together we get

Theorem 12. *The free boundary is locally $C^{m,\alpha}$ ($m \geq 2$, $0 < \alpha < 1$) or analytic, if the part of ∂U that parametrizes it is $C^{m+1,\alpha}$ or analytic. Any open part of the free boundary that has a reentrant corner as its p -closest point on the boundary, is analytic.*

Proof. Theorems 6 and 7 imply that d_p is $C^{m+1,\alpha}$ around the free boundary. Also by Theorem 9, the free boundary is locally a Jordan arc. Therefore we can apply Theorem 10 with $\phi = d_p$ and conclude that locally, the free boundary has a nondegenerate $C^{m,\alpha}$ parametrization. This together with the fact that the free boundary has no cusps (Theorem 11), give the result. The case of reentrant corners is similar, noting that d_p is analytic around them. \square

Note that by this theorem the free boundary is smooth, except at the finite number of points on it which have a reentrant corner as the p -closest point to them, and lie on the direction of an inward p -normal at that corner.

6. THE p -BISECTOR

Now we focus on understanding the shape of the free boundary. Consider the line $ax + by + c = 0$ and the point (x_0, y_0) in the plane. We want to find the p -distance of the point and the line. The direction of the p -normal to the line is

$$(a|a|^{\frac{2-p}{p-1}}, b|b|^{\frac{2-p}{p-1}}).$$

The p -closest point on the line to (x_0, y_0) , is the intersection of the line and the line that passes through (x_0, y_0) in the direction of the p -normal. Let that point be

$$(x_0, y_0) + t(a|a|^{\frac{2-p}{p-1}}, b|b|^{\frac{2-p}{p-1}}).$$

Then

$$a(x_0 + ta|a|^{\frac{2-p}{p-1}}) + b(y_0 + tb|b|^{\frac{2-p}{p-1}}) + c = 0.$$

Hence

$$(6.1) \quad t = -\frac{ax_0 + by_0 + c}{|a|^{\frac{p}{p-1}} + |b|^{\frac{p}{p-1}}}.$$

Therefore the p -distance is

$$(6.2) \quad \begin{aligned} d_p &= |t|(|a|^{\frac{p}{p-1}} + |b|^{\frac{p}{p-1}})^{\frac{1}{p}} \\ &= \frac{|ax_0 + by_0 + c|}{(|a|^{\frac{p}{p-1}} + |b|^{\frac{p}{p-1}})^{\frac{p-1}{p}}} \\ &= \frac{|ax_0 + by_0 + c|}{(|a|^q + |b|^q)^{\frac{1}{q}}}, \end{aligned}$$

where $q = \frac{p}{p-1}$ is the dual exponent to p . This also implies that

$$(6.3) \quad d_p = cd_2.$$

Where c is some positive constant depending only on p and the line, but not on the point (x_0, y_0) .

Definition 7. The p -bisector of an angle is the set of points inside the angle that have equal p -distance from each side of the angle.

It is easy to see from the above arguments that the p -bisector of an angle is a ray inside the angle emitting from its vertex.

7. NONREENTRANT CORNERS

Theorem 13. Suppose $w \in W^{1,\infty}(U)$, and $w \geq 0$ on ∂U .¹⁷ Also suppose that $-\Delta w \geq 0$ in the weak sense, i.e.

$$\int_U Dw \cdot D\phi \, dx \geq 0 \quad \text{for all } \phi \in H_0^1(U) \text{ with } \phi \geq 0 \text{ a.e..}$$

Then $w \geq 0$ on U .

¹⁷For this theorem, we assume that ∂U is Lipschitz.

Proof. Let $\phi = -w^- := -\min\{w, 0\}$. We have

$$-\int_{\{w < 0\}} |Dw|^2 dx = \int_U Dw \cdot D(-w^-) dx \geq 0.$$

As w is Lipschitz continuous, $\{w < 0\}$ is an open set. But by the above, $Dw = 0$ a.e. on this open set. Also $w = 0$ on $\partial\{w < 0\}$. Therefore $\{w < 0\}$ must be empty, and $w \geq 0$ everywhere on U . \square

Now suppose

$$(7.1) \quad G := \{(r, \theta) \mid 0 < r < r_0, -\alpha < \theta < \alpha\},$$

and $\alpha < \pi/2$.

Lemma 9. Suppose $u \in W^{1,\infty}(G)$ has Laplacian bounded from below in the weak sense, i.e. for some $C_1 > 0$ we have

$$\int_G Du \cdot D\phi dx \leq C_1 \int_G \phi dx \quad \text{for all } \phi \in H_0^1(G) \text{ with } \phi \geq 0 \text{ a.e..}$$

Also suppose that u vanishes on the straight sides of G . Then there are positive constants C, ν , such that for r small enough we have

$$(7.2) \quad u(r, \theta) \leq Cr^\nu d_p(r, \theta),$$

where d_p is the p -distance from the straight sides of G .

Proof. First assume that $\alpha > \pi/4$ and consider the following function on G

$$(7.3) \quad v := r^{\frac{\pi}{2\alpha}} \cos\left(\frac{\pi}{2\alpha}\theta\right) + \frac{r^2}{2}(\sin^2 \theta - \cos^2 \theta \tan^2 \alpha).$$

It is easy to see that in G

$$(7.4) \quad \Delta v = 1 - \tan^2 \alpha < 0,$$

and $v = 0$ on $\theta = \pm\alpha$. If r_0 is small enough then $v > 0$ on $r = r_0$. Also $D_\theta v(r_0, -\alpha) > 0$ and $D_{-\theta} v(r_0, \alpha) > 0$ (because $\cos(\frac{\pi}{2\alpha}\theta) \geq 0$ and $\frac{\pi}{2\alpha} < 2$). Hence for $C > 0$ large enough

$$(7.5) \quad \Delta(Cv - u) \leq 0 \quad \text{in the weak sense.}$$

And also $Cv - u > 0$ on $r = r_0$. Since its inward derivative is positive at $(r_0, \pm\alpha)$, so it is positive around them. (Actually, we only need to work with the Lipschitz constants.) Also on the remaining part of $r = r_0$ it is positive, because the coefficient of the dominant term of v there, i.e. $\cos(\frac{\pi}{2\alpha}\theta)$, has a positive minimum there. Also $Cv - u$ vanishes on $\theta = \pm\alpha$. Therefore by the maximum principle we have

$$(7.6) \quad u \leq Cv$$

inside G . Hence

$$u(r, \theta) \leq 2Cr^{\frac{\pi}{2\alpha}} \cos\left(\frac{\pi}{2\alpha}\theta\right) = 2Cr^\nu l \frac{\cos(\frac{\pi}{2\alpha}\theta)}{\omega(\theta)}.$$

Where $\nu = \frac{\pi}{2\alpha} - 1 > 0$, and

$$l = r\omega(\theta) = \begin{cases} r \sin(\theta + \alpha) & -\alpha < \theta < 0 \\ r \sin(\alpha - \theta) & 0 \leq \theta < \alpha \end{cases}$$

is the euclidean distance of the point (r, θ) from the straight sides of G . But the function $\frac{\cos(\frac{\pi}{2\alpha}\theta)}{\omega(\theta)}$ is bounded for $\theta \in (-\alpha, \alpha)$ as it is continuous and has finite limits at $\pm\alpha$. Thus

$$u(r, \theta) \leq \tilde{C}r^\nu l.$$

But by the formula (6.3) we have

$$d_p(r, \theta) \geq cl.$$

(Note that there are two sides and we have to take the minimum of both the distance and the p -distance to them, hence the equality in the formula (6.3) becomes an inequality.) Hence for small r we get

$$(7.7) \quad u(r, \theta) \leq \frac{\tilde{C}}{c} r^\nu d_p(r, \theta).$$

When $\alpha \leq \pi/4$, we need to find an appropriate replacement for v . Consider w , the minimizer of I over $K(G)$. As proved in the Remark 4, w satisfies the bound (7.2). It also has negative Laplacian around the vertex of G . Also, w vanishes on the straight sides of G , and is nonnegative inside it. Consider $\tilde{G} \subset G$, which consists of all points with $r < r_1$, where r_1 is small enough so that $\Delta w = -\eta$ on \tilde{G} . We can also choose r_1 such that w is positive on some point of $\partial\tilde{G}$. Since otherwise $w = 0$ on \tilde{G} and can not satisfy the equation. As $\Delta w < 0$ inside \tilde{G} and w is nonnegative on $\partial\tilde{G}$, strong maximum principle implies that $w > 0$ inside \tilde{G} .

Now let $\hat{G} \subset G$ be the set of points with $r < r_1 - \epsilon$. Then on the circular part of $\partial\hat{G}$ we have $w > 0$. Also the inward normal derivative of w is positive on the endpoints of this circular part. The reason is that it is nonnegative and if it was zero we would have $Dw = 0$ at that point (note that $w = 0$ on the straight sides of ∂G). But this gives a contradiction because the second derivative of w along the inward normal must be negative (since $\Delta w < 0$ and the second derivative of w vanishes on the straight sides of ∂G). Thus if we move along this inward normal w decreases and becomes negative which is a contradiction. Therefore w has all the necessary properties of v and the previous proof can be repeated (note that the comparison must be made on \hat{G} , not G). \square

Theorem 14. *If y_0 is a nonreentrant corner, then there is $R > 0$ such that $U \cap B_R(y_0)$ is elastic.¹⁸*

Proof. First suppose that y_0 is the intersection of two line segments S, S' of ∂U , making the angle $2\alpha < \pi$. We can choose a polar coordinate centered at y_0 such that

$$(7.8) \quad G := \{(r, \theta) \mid 0 < r < r_0, -\alpha < \theta < \alpha\} \subset U.$$

Note that

$$(7.9) \quad \Delta u = \begin{cases} -\eta & \text{in } E \\ \Delta d_p & \text{a.e. in } P, \end{cases}$$

and Δd_p is bounded away from the p -ridge especially in P .¹⁹ Thus we have that Δu is bounded in G . (Note that there is no reentrant corners there, so Δd_p does not blow up as we approach the boundary.) We assume that r_0 is small enough so that the p -closest point on ∂U to any point in G lies on the

¹⁸We need to assume that if one of the tangents to ∂U at y_0 is parallel to one of the coordinate axes, then Assumption 1 holds at y_0 , and around it there is no degenerate point of Assumption 2. This is necessary to ensure that κ_p is bounded around y_0 .

¹⁹Since κ_p is bounded around y_0 , $1 - \kappa_p d_p \rightarrow 1$ uniformly, as we approach ∂U . Also, $1 - \kappa_p d_p > 0$ on P around y_0 , so it has a positive minimum there. In addition, $\tau_p \kappa_p$ is bounded on ∂U as we assumed that S_i 's are smooth up to their endpoints. Hence Δd_p is bounded on P around y_0 .

sides of the angle at y_0 . Then Lemma 9 implies that the region $U \cap \{r < R\}$ is elastic for some small R , because

$$(7.10) \quad u(r, \theta) \leq Cr^\nu d_p(r, \theta) < CR^\nu d_p(r, \theta) < d_p(r, \theta).$$

Now suppose y_0 is a general nonreentrant corner. We reduce this case to the previous case. Let g be a conformal map from a sector G with straight sides to a neighborhood of y_0 (obviously the opening angle of G is the same as the one at y_0). As proved in Friedman [8], g and its inverse have bounded derivative up to the boundary. Let

$$(7.11) \quad \tilde{u}(z) := u(g(z))$$

be a function on G . Since g is conformal, we have

$$(7.12) \quad \Delta \tilde{u} = |Dg|^2 \Delta u.$$

And as $Dg, \Delta u$ are bounded, $\Delta \tilde{u}$ is also bounded. Hence by Lemma 9

$$\tilde{u} \leq C \tilde{d}_p \tilde{r}^\nu$$

near the vertex of G . Where \tilde{d}_p, \tilde{r} are respectively the p -distance from the straight sides of G , and the distance from its vertex. As Dg^{-1} is bounded we have $\tilde{d}_p \leq C d_p$ and $\tilde{r} \leq Cr$. Therefore we get $u < d_p$ in a neighborhood of y_0 . \square

Remark 4. This is another proof of the above theorem, when $\alpha \leq \pi/4$ and ∂U consists of line segments near y_0 . Here we only use Lemma 9 for $\alpha > \pi/4$.

Divide the angle by its p -bisector. It is enough to show that a sector of each of these smaller angles (the p -halves) is elastic. Let U' be a domain containing U such that $\partial U'$ contains S and another line segment S'' starting at y_0 on the same side of S as S' , making the angle $\gamma \in (\pi/2, \pi)$ with S . Thus y_0 is also a nonreentrant corner of $\partial U'$. Let w be the minimizer of I over $K(U')$. We claim that $u \leq w$ on U . To see this consider the test functions

$$v := u + (w - u)^- = \begin{cases} u & u \leq w \\ w & u > w \end{cases}$$

$$\tilde{v} := w + (u - w)^+ = \begin{cases} w & u \leq w \\ u & u > w \end{cases} = w - (w - u)^-.$$

(We consider u to be zero outside U .) Now u, w are both minimizers of the functional I over some closed convex sets. Also v, \tilde{v} belong to those sets respectively (note that $v = 0$ on ∂U as $w \geq 0 = u$ there). Hence we have

$$\begin{aligned} & \int_U Du \cdot D(v - u) - \eta(v - u) dx \\ &= \int_U Du \cdot D(w - u)^- - \eta(w - u)^- dx \geq 0, \end{aligned}$$

and

$$\begin{aligned} & \int_{U'} Dw \cdot D(\tilde{v} - w) - \eta(\tilde{v} - w) dx \\ &= \int_U -Dw \cdot D(w - u)^- + \eta(w - u)^- dx \geq 0. \end{aligned}$$

Note that $(w-u)^-$ is zero outside U as $w \geq 0$ and $u = 0$ there. Thus the domain of the second integral can be changed to U . Adding these two integrals we get

$$(7.13) \quad \int_U D(u-w) \cdot D(w-u)^- dx = - \int_{\{w < u\}} |D(w-u)|^2 dx \geq 0.$$

Therefore $w \geq u$ a.e. which by continuity gives the required result. Because otherwise we must have $w-u$ is a constant on the open set $\{w < u\}$ (again using continuity) and as it vanishes on the boundary of this set it must be zero on the set, which is a contradiction.

Hence for small r

$$u(r, \theta) \leq w(r, \theta) < d_p((r, \theta), \partial U').$$

But when the point (r, θ) is in the p -half attached to S of the angle at y_0 on ∂U (and r is small enough) we have

$$d_p((r, \theta), \partial U') = d_p((r, \theta), S) = d_p((r, \theta), \partial U).$$

So $u < d_p(\cdot, \partial U)$ as desired.

8. FLAT BOUNDARIES

Let $y(t)$ be a parametrization of part of ∂U and as before $y(t) + \delta(t)\mu(t)$ be the parametrization of the free boundary attached to this part. If $\delta(t) > 0$ for $t \in (a, b)$ and $\delta(a) = \delta(b) = 0$ then we call the set

$$\{y(t) + s\mu(t) \mid t \in [a, b], s \in [0, \delta(t)]\}$$

a **plastic component**.

Theorem 15. *The number of plastic components attached to a closed line segment²⁰ of ∂U is finite.*

Proof. For simplicity let the line segment be $\{(x_1, 0) \mid a \leq x_1 \leq b\}$. Suppose to the contrary that there are infinitely many plastic components

$$P_i = \{(x_1, x_2) \mid x_1 \in [a_i, b_i], x_2 \in [0, \delta(x_1)]\}$$

attached to the line segment. Where δ is a continuous nonnegative function on $[a, b]$ and $b_i \leq a_{i+1}$. Note that as we proved, δ is analytic on the set $\{\delta > 0\}$. Let

$$H_i := \max_{x \in [a_i, b_i]} \delta(x).$$

Since $b_i - a_i \rightarrow 0$ as $i \rightarrow \infty$, we must have $H_i \rightarrow 0$. Otherwise a subsequence, H_{n_i} converges to a positive number and by taking a further subsequence we can assume that this subsequence is $\delta(x_{n_i})$ where $x_{n_i} \rightarrow c$. But this contradicts the continuity of δ at c because $b_{n_i} \rightarrow c$ too.

Hence any line $x_2 = \epsilon$ intersects only a finite number $n(\epsilon)$ of P_i 's, and $n(\epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0$.

Consider a piecewise analytic curve γ in the elastic region E , such that it starts at $(a, 0)$ and ends at $(b, 0)$ (if P_i 's accumulate at one of these points, then γ will start or end at a point $(a - \epsilon, 0)$ or $(b + \epsilon, 0)$ if these points have an elastic neighborhood and if not, at the free boundary point on the lines $x_1 = a - \epsilon$ or $x_1 = b + \epsilon$). We also consider γ to be close enough to the segment so that for points between them the p -distance to ∂U is the p -distance to the segment, so for those points $d_p(x, \partial U)$ is a function of only x_2 .

Now consider E_ϵ to be the elastic region enclosed by γ and the line $x_2 = \epsilon$. The ∂E_ϵ consists of γ , segments on the line $x_2 = \epsilon$ and parts of ∂P_i 's. On every segment on $\partial E_\epsilon \cap \{x_2 = \epsilon\}$ with endpoints on the free boundary, the function $D_{x_1}(u - d_p) = D_{x_1}u$ is analytic and changes sign as $u - d_p$ is zero on

²⁰This means that the segment is a closed proper subset of a line segment of ∂U .

the endpoints and negative between them. Let (c, ϵ) , (d, ϵ) for $c < d$ be points close to those endpoints such that

$$D_{x_1}u(c, \epsilon) < 0, \quad D_{x_1}u(d, \epsilon) > 0.$$

Then as proved in Friedman [8] the level curves of the harmonic function $D_{x_1}u$ can be continued until they exit its domain $\cup E_\epsilon$. But $D_{x_1}u = D_{x_1}(u - d_p)$ is zero on the free boundary and on the segment $x_2 = 0$, hence they must exist through γ . This implies that $D_{x_1}u$ changes sign at least $2n(\epsilon) - 1$ times on γ , since the level curves do not cross each other. But this number grows to infinity as $\epsilon \rightarrow 0$ contradicting the fact that γ is piecewise analytic and $D_{x_1}u$ is analytic on a neighborhood of it. \square

Acknowledgement. I would like to express my gratitude to Lawrence C. Evans and Nicolai Reshetikhin for their invaluable help with this research.

REFERENCES

- [1] H. Brezis and M. Sibony. Équivalence de deux inéquations variationnelles et applications. *Arch. Rational Mech. Anal.*, 41:254–265, 1971.
- [2] L. A. Caffarelli and A. Friedman. The free boundary for elastic-plastic torsion problems. *Trans. Amer. Math. Soc.*, 252:65–97, 1979.
- [3] L. A. Caffarelli and N. M. Rivière. Smoothness and analyticity of free boundaries in variational inequalities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 3(2):289–310, 1976.
- [4] L. A. Caffarelli and N. M. Rivière. The smoothness of the elastic-plastic free boundary of a twisted bar. *Proc. Amer. Math. Soc.*, 63(1):56–58, 1977.
- [5] L. A. Caffarelli, A. Friedman, and G. Pozzi. Reflection methods in the elastic-plastic torsion problem. *Indiana Univ. Math. J.*, 29(2):205–228, 1980.
- [6] G. Crasta and A. Malusa. The distance function from the boundary in a Minkowski space. *Trans. Amer. Math. Soc.*, 359(12):5725–5759 (electronic), 2007.
- [7] D. De Silva and O. Savin. Minimizers of convex functionals arising in random surfaces. *Duke Math. J.*, 151(3):487–532, 2010.
- [8] A. Friedman. *Variational Principles And Free-Boundary Problems*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1982. A Wiley-Interscience Publication.
- [9] A. Friedman and G. Pozzi. The free boundary for elastic-plastic torsion problems. *Trans. Amer. Math. Soc.*, 257(2):411–425, 1980.
- [10] Y. Li and L. Nirenberg. The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations. *Comm. Pure Appl. Math.*, 58(1): 85–146, 2005.
- [11] R. T. Rockafellar. *Convex Analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.
- [12] M. Safdari. The regularity of some vector-valued variational inequalities with gradient constraints. *preprint*.
- [13] G. Treu and M. Vornicescu. On the equivalence of two variational problems. *Calc. Var. Partial Differential Equations*, 11(3):307–319, 2000.

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA
safdari@berkeley.edu